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Note

A bound on the total size of a cut cover

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Abstract

A *cycle cover* (*cut cover*) of a graph G is a collection of cycles (cuts) of G that covers every edge of G at least once. The *total size* of a cycle cover (cut cover) is the sum of the number of edges of the cycles (cuts) in the cover.

We discuss several results for cycle covers and the corresponding results for cut covers. Our main result is that every connected graph on n vertices and e edges has a cut cover of total size at most $2e - n + 1$ with equality precisely when every block of the graph is an odd cycle or a complete graph (other than K_4 or K_8). This corresponds to the result of Fan [J. Combin. Theory Ser. B 74 (1998) 353–367] that every graph without cut-edges has a cycle cover of total size at most $e + n - 1$.

Keywords: Cut; Cycle; Edge cover

1. Cycle covers and cut covers

Covering the edges of a graph by subgraphs from a given family of graphs, like cliques, matchings, trees, cycles, or cuts is one of the basic themes in graph theory (see Pyber [21] for a survey of results). Erdős et al. [5] showed that the edges of every graph on n vertices can be covered by $\lfloor n^2/4 \rfloor$ cliques, and the balanced complete bipartite graph shows that this is best possible. It can also be desirable to minimize parameters other than the number of subgraphs used in the cover. Győri and Kostochka [11], Chung [4] and Kahn [17] independently proved the stronger result that every graph has a decomposition into

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cliques whose *order-sum* (sum of the number of vertices of the cliques in the cover) is at most $\lfloor n^2/2 \rfloor$.

A heavily studied edge covering concept is that of a cycle cover: A *cycle cover* of a graph G is a collection of cycles of G such that every edge of G is in at least 1 cycle. Since an edge of a graph is in a cycle precisely when it is not a cut-edge, cycle covers exist only for graphs without cut-edges (also called *bridgeless* graphs). One of the outstanding questions on cycle covers is the Cycle Double Cover Conjecture (CDCC) of Seymour [22] and Szekeres [23]: Every bridgeless graph has a cycle cover such that every edge is in exactly 2 cycles. This question is connected to several topological questions. The interested reader should consult the book of Zhang [26] for more information on the CDCC and other cycle cover problems.

The *total size* of a cycle cover is the sum of the number of edges over all cycles in the cover. Thus, a cycle double cover of a graph on e edges has total size $2e$. The minimum total size of a cycle cover of a bridgeless graph G is denoted by $\mathrm{scc}(G)$ (for shortest cycle cover). There are a number of interesting questions on shortest cycle covers of graphs. In 1995, Thomassen [24] settled a conjecture of Itai et al. [12] by proving that the problem of determining $\mathrm{scc}(G)$ is NP-complete. A conjecture of Alon and Tarsi [1] claims that every bridgeless graph on e edges satisfies $\mathrm{scc}(G) \leqslant \frac{7}{5}e$ with equality for the Petersen graph and various graphs derived from it. Interestingly, Jamshy and Tarsi [16] proved that this conjecture implies the CDCC.

A *cut* induced by a set of vertices S consists of all edges with exactly one endpoint in S. The notions of a *cut cover* of a graph G, and the minimum total size of such a cut cover, denoted by $\mathrm{ccs}(G)$ (for cut cover size) can now be defined similarly. For a bridgeless graph G embedded in the plane, the dual of a cycle forms a cut in the dual graph G^*, so that it is easy to see that $\mathrm{scc}(G) = \mathrm{ccs}(G^*)$. Considering this duality it seems reasonable to expect that similarly intriguing questions arise when studying cut covers. It turns out that in several ways cuts behave nicer than cycles. For one, every graph *has* a cut cover.

A *star cut* is a cut in which the set S has size 1. This notion leads immediately to a "Cut Double Cover Theorem": if a cut cover consists of all the star cuts of a graph, then every edge is covered exactly twice. Hence $\mathrm{ccs}(G) \leqslant 2e(G)$. Unfortunately, however, determining $\mathrm{ccs}(G)$ is still an NP-complete problem, even when restricted to graphs with maximum degree 3 (see [10]).

The focus of this paper is the dual question to the following cycle cover question of Itai and Rodeh [13]: Does every bridgeless graph on n vertices and e edges have a cycle cover of total size at most $e + n - 1$, i.e. is $\mathrm{scc}(G) \leqslant e + n - 1$? After a flurry of papers [13,12,1,3,9,6,7,2] this question was finally settled in the affirmative by Fan [8]. By Euler's formula we thus obtain that for a (loopless) connected plane graph with n vertices, e edges and f faces

$$\mathrm{ccs}(G) = \mathrm{scc}(G^*) \leqslant e + f - 1 = e + (2 + e - n) - 1 = 2e - n + 1.$$

Our main result is that this bound holds for non-planar graphs as well and we characterize the cases of equality:

Theorem 1. *If G is a connected graph on e edges and n vertices, then*

$$\mathrm{ccs}(G) \leqslant 2e - n + 1.$$

Equality holds if and only if every block of G is an odd cycle or a complete graph other than K_4 or K_8.

The condition that G is connected is crucial, since otherwise $2e-n+1$ could be negative. However the following result follows easily by induction on k:

Corollary 2. *If G is a graph on e edges, n vertices, and k components, then*

$$\mathrm{ccs}(G) \leqslant 2e - n + k.$$

Equality holds if and only if every block of G is an odd cycle or a complete graph other than K_4 or K_8.

Furthermore using the fact that $e \leqslant \binom{n}{2}$ one easily obtains the dual version of the Alon–Tarsi conjecture (for a different proof see [20]).

Corollary 3. *If G is a simple graph on e edges, then*

$$\mathrm{ccs}(G) \leqslant 2e - \sqrt{2e + \frac{1}{4}} + \frac{1}{2}.$$

Equality holds when $e = \binom{n}{2}$ for some $n \neq 4, 8$.

Returning to cycle cover questions, we observe that Theorem 1 yields a new proof of Fan's theorem for planar graphs. Furthermore, since K_n is non-planar for $n > 4$, equality is only achieved by graphs obtained from trees by replacing each edge by an odd number of parallel edges (and maybe adding some loops). We mention a simple self-contained proof of Fan's result for planar graphs in Theorem 4.

2. Definitions and related results

We largely follow the notation of West [25]. Throughout this paper G is a loopless graph with vertex set $V = V(G)$, and edge set $E = E(G)$. If we want to specify that G has no parallel edges, then we call G *simple*. For a given graph G we define its *order* by $n = n(G) = |V(G)|$, and its *size* by $e(G) = |E(G)|$. For a partition of the vertex set $V = S \cup \overline{S}$ we define the *cut* induced by S to be the set of edges between S and $\overline{S}$,

$$[S, \overline{S}] := \{uv \in E(G) : u \in S, v \in \overline{S}\}.$$

If $|S| = 1$, then $[S, \overline{S}]$ is a *star cut*. A *cut cover* of a graph G is a collection $\mathscr{C} = \{[S_1, \overline{S}_1], [S_2, \overline{S}_2], \ldots, [S_k, \overline{S}_k]\}$ of cuts whose union is $E(G)$. The *total size* of $\mathscr{C}$ is the sum of the sizes $|[S_i, \overline{S}_i]|$ of the cuts in $\mathscr{C}$. The *cut cover size* of G, denoted by $\mathrm{ccs}(G)$ is the minimum total size of a cover of $E(G)$ with cuts.

We immediately get the trivial bounds that $e(G) \leqslant \mathrm{ccs}(G) \leqslant \mathrm{ccs}(K_n)$ for simple graphs, where equality in the lower bound holds for all bipartite graphs. The cut cover size of the complete graph has been determined in [14,15,18,19]:

$$\mathrm{ccs}(K_n) = \begin{cases} (n-1)^2, & n \neq 4, 8, \\ (n-1)^2 - 1, & n = 4, 8. \end{cases} \tag{1}$$

For complete graphs with at least 8 vertices the optimal cut cover is unique, up to isomorphism. For $n > 8$, cover K_n with $n-1$ star cuts. For $n = 8$, cover K_8 with $K_{4,4}$'s by taking 3 cuts such that $|S_i| = 4$, $|S_i \cap S_j| = 2$ for $i \neq j$ and $|S_1 \cap S_2 \cap S_3| = 1$.

For odd cycles we have

$$\mathrm{ccs}(C_{2k+1}) = 2k + 2. \tag{2}$$

Indeed, every cut in a cycle has even size, so $\mathrm{ccs}(C_{2k+1})$ must also be even. Together with the trivial lower bound, this fact yields the lower bound. There are many different covers achieving this value.

A vertex set S is *stable*, or *independent*, if the subgraph induced by S has no edges. A *stable cut* $[S, \overline{S}]$ is a cut in which S is a stable set. The maximum size of a stable cut is denoted by

$$\mathrm{Cut}'(G) := \max\{|[S, \overline{S}]| : S \subset V, \ \ S \text{ stable set}\}. \tag{3}$$

If $[S, \overline{S}]$ is a stable cut of G, then we can obtain a cut cover of G by using star cuts induced by the vertices $v \in \overline{S}$. As observed in [10] the edges in $[S, \overline{S}]$ are covered once, and all other edges twice, so that

$$\mathrm{ccs}(G) \leqslant 2e(G) - \mathrm{Cut}'(G). \tag{4}$$

3. Fan's theorem for planar graph

We start with a simple proof of Fan's theorem [8] for planar graphs which is essentially dual to our proof of Theorem 1.

Theorem 4. *If G is a* 2 *edge-connected plane graph on e edges and n vertices, then there is cycle cover of G of total length at most* $e + n - 1$ *which uses only facial cycles.*

Proof. We may assume that the f faces of G are ordered $F_1, F_2, \ldots, F_f$ such that every F_i (for $i \geqslant 2$) shares an edge with one of the faces preceding it. We will now recursively partition $F_1, \ldots, F_i$ into an independent set $\mathscr{I}_i$ (no two faces in $\mathscr{I}_i$ have a common edge) and a collection of faces $\mathscr{C}_i$ such that there are at least $i-1$ edges each of which is in a face in $\mathscr{I}_i$ as well as $\mathscr{C}_i$.

To start, let $\mathscr{I}_1 = \{F_1\}$ and $\mathscr{C}_1 = \emptyset$. Suppose we have successfully constructed $\mathscr{I}_i$ and $\mathscr{C}_i$ which have $i-1$ common edges. Consider F_{i+1}. If F_{i+1} shares an edge with a face in $\mathscr{I}_i$, then let $\mathscr{I}_{i+1} = \mathscr{I}_i$ and $\mathscr{C}_{i+1} = \mathscr{C}_i \cup \{F_{i+1}\}$ and observe that these sets now share at least one more common edge. Otherwise $\mathscr{I}_{i+1} = \mathscr{I}_i \cup \{F_{i+1}\}$ and $\mathscr{C}_{i+1} = \mathscr{C}_i$, so that by the ordering of the faces again these sets share at least one more common edge.

After considering all f faces, $\mathscr{I}_f$ and $\mathscr{C}_f$ have at least $f-1$ common edges. Since G is 2 edge connected, the boundaries of the faces in $\mathscr{C}_f$ form cycles, and by the definition of $\mathscr{I}_f$ these cycles cover all edges of G. Moreover, the total length of the cover is at most $2e - (f - 1) = e + n - 1$, by Euler's formula. $\square$

4. Proof of Theorem 1

For a graph G on n vertices let $\mathrm{def}(G) = \mathrm{Cut}'(G) - n + 1$ denote the *deficit* of G. Since we can write (4) as

$$\mathrm{ccs}(G) \leqslant 2e(G) - n(G) + 1 - \mathrm{def}(G) \tag{5}$$

we would like to prove that every connected graph has deficit at least zero.

Lemma 5. *If H is an induced subgraph of a connected graph G, then*

$$\mathrm{def}(G) \geqslant \mathrm{def}(H).$$

Proof. We may assume that $H \neq G$. Let $[S, \overline{S}]$ be a maximum stable cut of H. Since G is connected there must be a vertex $v \notin V(H)$ which has a neighbor in $V(H)$. If v has no neighbor in S, then $[S+v, \overline{S}]$ is a stable cut which establishes that $\mathrm{Cut}'(H+v) \geqslant \mathrm{Cut}'(H)+1$, so that $\mathrm{def}(H + v) \geqslant \mathrm{def}(H)$. If v has a neighbor in S, then $[S, \overline{S} + v]$ similarly shows that $\mathrm{def}(H + v) \geqslant \mathrm{def}(H)$. Repeating this process it follows that $\mathrm{def}(G) \geqslant \mathrm{def}(H)$. $\square$

A *theta graph* is a simple graph which consists of 3 internally vertex-disjoint paths between two distinguished vertices u, v. The smallest theta graph is K_4 with any edge removed. Observe that every theta graph contains an even cycle.

We can now prove Theorem 1 for 2-connected graphs:

Theorem 6. *If G is a 2-connected graph on e edges and n vertices, then*

$$\mathrm{ccs}(G) \leqslant 2e - n + 1.$$

Equality holds if and only if G is an odd cycle or a complete graph other than K_4 or K_8.

Proof. To prove the inequality it suffices by (5) to show that $\mathrm{def}(G) \geqslant 0$. This follows immediately from Lemma 5 since G must have K_1 as a subgraph, and thus $\mathrm{def}(G) \geqslant \mathrm{def}(K_1) = 0$. Furthermore, if G is a complete graph (for $n \neq 4, 8$) or an odd cycle, then a quick calculation shows that equality must hold by Eqs. (1) and (2).

It remains to show the “only if” part of the second assertion of the theorem. So let G be a 2-connected graph with $\mathrm{ccs}(G) = 2e - n + 1$, and thus $\mathrm{def}(G) = 0$. We must show that G is either an odd cycle or a complete graph (for $n \neq 4, 8$). By Lemma 5 it follows that every induced subgraph H of G must satisfy $\mathrm{def}(H) \leqslant 0$. So since a two vertex graph H has $\mathrm{def}(H) = e(H) - 1$ it follows that G must be a simple graph.

Observe that every even cycle C of G must have at least two chords, since otherwise the bipartition of C would yield a stable cut of the subgraph H induced by $V(C)$, so that $\mathrm{def}(G) \geqslant \mathrm{def}(H) \geqslant e(C) - n(H) + 1 = 1$.

Claim 1. *Every even cycle C_{2k} of G induces a complete subgraph.*

By the previous observation this certainly holds for C_4 and we can proceed by induction on $k \geqslant 2$. So let $v_1, v_2, \ldots, v_{2k}$ form an even cycle C.

We first show that if C contains a chord of the form $v_i v_j$ where i is odd and j is even (call this an *even chord*), then $V(C)$ induces a complete graph. Observe that $v_i v_j$ splits C into two smaller even cycles C^1, C^2 both of which must thus induce a complete graph. To see that C itself induces a complete graph consider two arbitrary vertices $v_a \in V(C^1) - V(C^2)$ and $v_b \in V(C^2) - V(C^1)$. Since $v_i v_a v_j v_b$ yields a C_4, the edge $v_a v_b$ must be in G.

By observation C_{2k} must contain at least 2 chords, and we can assume without loss of generality that they are $v_1 v_m$ and $v_i v_j$ for $1 < i < j, m$. We may also assume that m is odd, and i, j have the same parity. Now if $j \leqslant m$, then $v_1, v_2, \ldots, v_i, v_j, v_{j+1}, \ldots, v_m$ forms a shorter even cycle, and thus induces a complete subgraph. Hence $v_2 v_m$ is an even chord of C and we are done. So suppose $m < j \leqslant 2k$. If $i = 2$ and $j = m + 1$, then $v_1 v_i v_j v_m$ forms a 4-cycle, so that again $v_2 v_m$ is an even chord. Otherwise $v_1, v_{2k}, \ldots, v_j, v_i, v_{i+1}, \ldots, v_m$ forms an even cycle shorter than C, so that either $v_1 v_{m-1}$ is an even chord (for $m > 3$) or $v_m v_{2k}$ is an even chord. This finishes the proof of Claim 1.

Claim 2. *Every theta subgraph H of G induces a complete subgraph.*

Since H must contain an even cycle C, C induces a complete subgraph by Claim 1. If $x \in V(H) - V(C)$, then C contains vertices u, v such that H consists of C and a u, v-path P containing x. If $|V(H)|$ is even, then P followed by the remaining vertices of C forms an even cycle C' in G and thus $V(H) = V(C')$ induces a complete subgraph by Claim 1. If $|V(H)|$ is odd, then P followed by all but one vertex y of C forms an even cycle, and thus a complete graph. The remaining edges from y can be established by leaving out another vertex y' of C instead.

Now that both claims have been established, consider a longest cycle C of G. (Since G is 2-connected, such a cycle must exist.) To see that C must be spanning (that is Hamiltonian), consider any $v \notin V(C)$. Since G is 2-connected, there are 2 internally disjoint paths from v to $V(C)$. These paths together with C form a theta graph. By Claim 2, this theta graph induces a complete subgraph and we can find a cycle longer than C, contradicting the maximality of C.

So C is a spanning cycle. If n is even, then by Claim 1, $V(G) = V(C)$ must induce a complete graph. If n is odd, then either G is an odd cycle, or C must contain a chord uv. In the latter case $C + uv$ forms a theta graph, so that $V(G) = V(C)$ must induce a complete graph by Claim 2.

Hence either G is an odd cycle or a complete graph. However, G cannot be K_4 or K_8, since by (1) these do not satisfy $\mathrm{ccs}(G) = 2e + n - 1$. $\square$

The proof of Theorem 1 now follows with the help of the following lemma.

Lemma 7. *Let G be a graph with a cut-vertex v and G_1, G_2 be non-trivial subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$.*

(1) $e(G) = e(G_1) + e(G_2)$,
(2) $n(G) = n(G_1) + n(G_2) - 1$ *and*
(3) $\mathrm{ccs}(G) = \mathrm{ccs}(G_1) + \mathrm{ccs}(G_2)$.

Proof. The first two equations are immediate. For the third equation observe that a cut cover of G yields a simultaneous cut cover of G_1 and G_2. Thus $\mathrm{ccs}(G) = \mathrm{ccs}(G_1 \cup G_2) \geqslant \mathrm{ccs}(G_1) + \mathrm{ccs}(G_2)$.

Every cut $[S, \overline{S}]$ in G_1 which has $v \in S$ can be converted to a cut $[S \cup V(G_2), \overline{S}\,]$ of G such that $[S \cup V(G_2), \overline{S}\,] = [S, \overline{S}]$. A similar argument works for G_2. Since we may assume that without loss of generality $v \in S_i$ for every cut $[S_i, \overline{S}_i]$ in a cut cover of G_1 (G_2), we can hence form a cut cover for G from cut covers for G_1 and G_2. Thus $\mathrm{ccs}(G) \leqslant \mathrm{ccs}(G_1) + \mathrm{ccs}(G_2)$. $\square$

Proof of Theorem 1. Observe that as in the proof of Theorem 6, $\mathrm{def}(G) \geqslant \mathrm{def}(K_1) = 0$ so that the inequality follows. To characterize the cases of equality we proceed by induction on n with trivial base cases $G = K_1, K_2$.

So let G be a graph on $n \geqslant 3$ vertices and e edges. If G is 2-connected, then we are done by Theorem 6. Otherwise G contains a cut-vertex v, and connected graphs G_1, G_2 as in Lemma 7. Observe that every block of G is a block of G_1 or G_2 and vice versa.

Hence, if the blocks of G are odd cycles or complete graphs other than K_4 or K_8, then the same must hold for G_1 and G_2. Thus by hypothesis $\mathrm{ccs}(G_1) = 2e(G_1) - n(G_1) + 1$ and $\mathrm{ccs}(G_2) = 2e(G_2) - n(G_2) + 1$, so that $\mathrm{ccs}(G) = \mathrm{ccs}(G_1) + \mathrm{ccs}(G_2) = (2e(G_1) - n(G_1) + 1) + (2e(G_2) - n(G_2) + 1) = 2e - n + 1$.

Conversely, if $\mathrm{ccs}(G) = 2e - n + 1$, then $\mathrm{ccs}(G_1) + \mathrm{ccs}(G_2) = \mathrm{ccs}(G) = 2e - n + 1 = (2e(G_1) - n(G_1) + 1) + (2e(G_2) - n(G_2) + 1)$. However, since $\mathrm{ccs}(G_1) \leqslant 2e(G_1) - n(G_1) + 1$ and $\mathrm{ccs}(G_2) \leqslant 2e(G_2) - n(G_2) + 1$ it follows that equality must hold in both inequalities. Thus by hypothesis the blocks of G_1 and G_2 are odd cycles or complete graphs other than K_4 or K_8 and thus the same must hold for G. $\square$

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